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Approximation by Exponential Sums on Discrete and Continuous Domains*

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In this paper we develop an existence theory for approximation from an enlargement of the set of exponential sums, $V_n(S)$, where S is a closed subset of $\mathbb{R}$. Here $V_n(S)$ denotes the set of all solutions of all n-th order homogeneous differential equations with constant coefficients for which the roots of the corresponding characteristic polynomial all lie in the set S. Our theory encompasses a wide class of norms including the customary L_p norms, $1 \leqslant p \leqslant \infty$, on intervals of the type $[0, b]$, $0 < b \leqslant +\infty$, and the familiar l_p norms, $1 \leqslant p \leqslant \infty$, on (possibly infinite) discrete point sets where the points are not necessarily uniformly spaced.

1. INTRODUCTION

Let $S \subseteq \mathbb{R}$ where $\mathbb{R}$ is the set of real numbers. We define $V_n(S)$, $n = 1, 2, \ldots$, to be the set of all real valued functions y defined on $\mathbb{R}$ which satisfy some nth order homogeneous differential equation of the form

$$[(D + \lambda_1) \cdots (D + \lambda_n)]\, y(t) \equiv 0, \qquad -\infty < t < \infty \tag{1}$$

where D is the differential operator d/dt and where $\lambda_1, \ldots, \lambda_n \in S$. We define $V_0(S)$ to be the set whose only element is the zero function and we set

$$V_\infty(S) = \bigcup_{n=1}^{\infty} V_n(S).$$

The existence of a best approximation from $V_n(\mathbb{R})$ to a given continuous real valued function defined on D, $D \subseteq \mathbb{R}$, using the uniform norm

$$\| F \| = \sup\{| F(t)|: t \in D\}$$

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has been widely studied. Indeed, Braess [1], Rice [16], Kammler [8], and Werner [20, 21] have all shown that a best approximation does exist when the domain D is a compact interval. Braess [1, 2], Rice [17], Rosman [18] and Loeb and Wolfe [13] have investigated the existence question when D consists of a finite number of uniformly spaced points. Additionally, Kammler [8, 10] has studied the existence of best approximations to L_p-functions on intervals of the type $[0, b]$, $0 < b \leqslant +\infty$, using the customary L_p norms, $1 \leqslant p \leqslant \infty$.

In this paper we develop an existence theory which encompasses a wide class of norms, including the norms employed in the above papers. More importantly, our theory also handles the significant case where the domain of approximation is a (possibly infinite) discrete point set where the points are not necessarily uniformly spaced.

Let σ be a nondecreasing, left-continuous, real valued function on $\mathbb{R}$ satisfying $\sigma(t) = 0$ for $t \leqslant 0$. The point $t \in \mathbb{R}$ is a point of increase of σ if there is no neighborhood of t such that σ is constant on that neighborhood. Clearly, the set of points of increase of σ, which we denote by D_σ, is a closed set. We denote the Lebesgue-Stieltjes measure associated with σ by λ_σ (cf. [7, pp. 330–331]), and so $\lambda_\sigma(A)$ is the λ_σ-measure of the set $A \subseteq \mathbb{R}$. We define the L_p spaces associated with λ_σ and their corresponding norms in the usual manner, and we denote them by $\mathscr{L}_{p,\sigma}$, $1 \leqslant p \leqslant \infty$, and $\| \ \|_{p,\sigma}$, $1 \leqslant p \leqslant \infty$, respectively.

Given σ, $S \subseteq \mathbb{R}$, $1 \leqslant p \leqslant \infty$, and $f \in \mathscr{L}_{p,\sigma}$ we would like to find a best $\| \ \|_{p,\sigma}$-approximation to f from $V_n(S)$, i.e., we would like to find a $y_0 \in V_n(S)$ such that

$$\|f - y_0\|_{p,\sigma} = \inf\{\|f - y\|_{p,\sigma} : y \in V_n(S)\}. \tag{2}$$

For example, if $0 < b \leqslant +\infty$ and if

$$\begin{aligned} \sigma(t) &= 0, && \text{if} \quad t \leqslant 0 \\ &= t, && \text{if} \quad 0 < t < b \\ &= b, && \text{if} \quad t \geqslant b \end{aligned} \tag{3}$$

then the $\mathscr{L}_{p,\sigma}$-norm reverts to the usual L_p-norm on $[0, b]$ (i.e., λ_σ agrees with the Lebesgue measure on $[0, b]$), and when S is any closed subset of $\mathbb{R}$ we know from prior work [8, Theorem 2; 10, Theorem 3] that every $\mathscr{L}_{p,\tau}$ function, $1 \leqslant p \leqslant \infty$ has a best $\| \ \|_{p,\sigma}$-approximation from $V_n(S)$, $n = 0, 1,\ldots$. On the other hand, when σ is given by

$$\begin{aligned} \sigma(t) &= 0, && \text{if} \quad t \leqslant 0 \\ &= i, && \text{if} \quad (i-1)/N < t \leqslant i/N, \quad i = 1,\ldots, N \\ &= N + 1, && \text{if} \quad t > 1 \end{aligned} \tag{4}$$

then the $\mathscr{L}_{p,\sigma}$-norm is the customary ℓ_p norm on D_σ and it is well known (cf. [17, pp. 65–69]) that a best ℓ_p-approximation, $1 \leqslant p \leqslant \infty$, from $V_n(\mathbb{R})$ to a function defined on a finite point set need not exist. However, when S is restricted to be a compact interval, Braess [1, Satz 5] has shown that a best ℓ_∞-approximation from $V_n(S)$ to a function defined on a finite domain does exist.

The nonexistence of a best approximation is not restricted to finite point sets.

EXAMPLE 1. If we define σ so that

$$\begin{aligned} \sigma(t) &= 0, && \text{if } t \leqslant 0 \\ &= 1 + 2^{-n}, && \text{if } 2^{-n} < t \leqslant 2^{-n+1}, \quad n = 1, 2,... \\ &= 2, && \text{if } t > 1 \end{aligned}$$

then any function satisfying

$$\begin{aligned} f(t) &= 1, && t = 0 \\ &= 0, && t \in D_\sigma \backslash \{0\} \end{aligned}$$

has no best $\| \ \|_{p,\sigma}$-approximation from $V_1(\mathbb{R})$, $1 \leqslant p < \infty$. Indeed, if $y_\nu(t) = \exp(-\nu t)$, $\nu = 1, 2,...$ then

$$\| f - y_\nu \|_{p,\sigma} = \left\{ \sum_{n=1}^{\infty} [\exp(-\nu \cdot 2^{-n+1})]^p \cdot 2^{-n} \right\}^{1/p}$$

and simple estimates show that

$$\inf\{\| f - y \|_{p,\sigma} : y \in V_1(\mathbb{R})\} \leqslant \lim \| f - y_\nu \|_{p,\sigma} = 0.$$

However, the exponential sum $y(t) \equiv 1/2$ is a best $\| \ \|_{\infty,\sigma}$-approximation to f from $V_1(\mathbb{R})$. ∎

We thus see that the existence of a solution to (2) depends on the choice of σ, f, S, and n. However, when S is closed, we shall show how to enlarge (depending upon σ, n, and S) the class of approximating functions, $V_n(S)$, so as to obtain an existence theory. We first formulate the approximating class and then develop the existence theory for the case when D_σ is compact. Subsequently, we extend the existence theory to the case where D_σ is not compact.

As one might suspect, there are functions for which no enlargement of the approximating class is needed. When $p = \infty$ we shall show that the class of completely monotone functions is such a collection of functions.

2. Enlarged Approximating Class

In order to correctly expand the approximating class $V_n(\mathbb{R})$ we must account for the behavior typified by the approximating sequence in Example 1, i.e., we must include those functions which are suitable linear combinations of exponential sums and functions which have support consisting of appropriately chosen "endpoints" of D_σ. (A function g has support D if $g(t) = 0$ for $t \notin D$.)

Before proceeding we first impose the following assumption on σ.

∞-*Assumption.* When $p = \infty$ we shall assume that σ is continuous at the largest and smallest accumulation points of D_σ.

Indeed, suppose that σ is given, that t_ℓ, t_r are the leftmost, rightmost accumulation points of D_σ, respectively, that $j_\ell = \sigma(t_\ell+) - \sigma(t_\ell)$, and that $j_r = \sigma(t_r+) - \sigma(t_r)$. If we then define σ^* so that

$$\begin{aligned} \sigma^*(t) &= \sigma(t), && \text{if} \quad t \leqslant t_\ell \\ &= \sigma(t) - j_\ell, && \text{if} \quad t_\ell < t \leqslant t_r \\ &= \sigma(t) - j_\ell - j_r, && \text{if} \quad t > t_r \end{aligned}$$

then $D_\sigma = D_\sigma^*$ and when f is continuous we also have $\|f\|_{\infty,\sigma} = \|f\|_{\infty,\sigma}^*$ so that no generality is lost in the important case where f is continuous.

Let σ be given and let J_σ be the set of points of (jump) discontinuity of σ. We define sets $L_{n,\sigma}$, $n = 0, 1,...$ inductively as follows. We take $L_{0,\sigma} = \varnothing$. For $n = 1, 2,...$ we let $t_n = \inf(D_\sigma \backslash L_{n-1,\sigma})$. If $t_n \in J_\sigma$ then $L_{n,\sigma} = L_{n-1,\sigma} \cup \{t_n\}$. Otherwise, $L_{n,\sigma} = L_{n-1,\sigma}$. Hence, $L_{n,\sigma}$ is the intersection of J_σ with the largest possible collection of n or fewer "leftmost" points of D_σ. In a similar manner, we take $R_{n,\sigma}$ to be the intersection of J_σ with the largest possible collection of n or fewer "rightmost" points of D_σ.

Let σ be given, let $S \subseteq \mathbb{R}$, and let n be a nonnegative integer. We define our enlarged approximating class, $P_{n,\sigma}(S)$, to be the set of all functions q of the form

$$q(t) = y(t) + \ell(t) + r(t), \qquad t \in \mathbb{R}$$

where $y \in V_{n_1}(S)$, where ℓ is a real valued function with support $L_{n_2,\sigma}$, where r is a real valued function with support $R_{n_3,\sigma}$, and where n_1, n_2, n_3 are nonnegative integers with $n_1 + n_2 + n_3 \leqslant n$, with $n_2 = 0$ if S is bounded below, and with $n_3 = 0$ if S is bounded above.

Given σ, $1 \leqslant p \leqslant \infty$, $f \in \mathscr{L}_{p,\sigma}$, and a nonnegative integer n we shall establish the existence of a best $\| \ \|_{p,\sigma}$-approximation to f from $P_{n,\sigma}(S)$ under the assumption that S is a given closed subset of $\mathbb{R}$.

Our theory is consistent with existing work. Indeed, when σ is given by (3), so that $D_\sigma = [0, b]$ and $L_{n,\sigma} = R_{n,\sigma} = \varnothing$ for each $n = 0, 1,...,$

and when S is a subset of $\mathbb{R}$, then $P_{n,\sigma}(S) = V_n(S)$. As we have observed, when S is closed then every $f \in \mathscr{L}_{p,\sigma}$, $1 \leqslant p \leqslant \infty$, has a best $\mathscr{L}_{p,\sigma}(= L_p)$-approximation from $V_n(S)$. Furthermore, in the special case where σ is a step function with a finite number of uniformly spaced points of discontinuity, i.e., when σ is given by (4), so that $D_\sigma = \{i/N: i = 0,..., N\}$, $L_{n,\sigma} = \{(i-1)/N: i = 1,..., n\} \cap [0, 1]$, and $R_{n,\sigma} = \{(N+1-i)/N: i = 1,..., n\} \cap [0, 1]$, then our enlarged approximating class of functions, $P_{n,\sigma}(\mathbb{R})$, agrees with that introduced by Loeb and Wolfe [13], i.e., $P_{n,\sigma}(\mathbb{R})$ contains all functions which have the form $y(t) + f(t)$ where $y \in V_{n_1}(\mathbb{R})$ and where f is an arbitrary real valued function which has support $\{0, 1/N,..., (n_2 - 1)/N, (N + 1 - n_3)/N,..., 1\}$. As is shown in [13], every real valued function defined on D_σ has a best $\mathscr{L}_{p,\sigma}(= \ell_p)$-approximation from $P_{n,\sigma}(\mathbb{R})$, $1 \leqslant p \leqslant \infty$, $n = 0, 1,...$.

3. Compact Domain

We first develop our existence theory under the assumption that D_σ (and thus the effective domain of f) is compact. We shall further assume that D_σ has at least two points so that after effecting a translation and a scale change, if necessary, we may further assume that $D_\sigma \subseteq [0, 1]$ with $0, 1 \in D_\sigma$.

We begin by adopting some notation that closely follows that introduced in [8]. If y satisfies (1) but does not satisfy any such differential equation of lower order we shall say that y is an exponential sum with order n. The n (not necessarily distinct) real numbers $\lambda_1,..., \lambda_n$ are then called the essential exponential parameters of y, and we shall refer to the set

$$\Lambda[y] = \bigcup_{i=1}^{n} \{\lambda_i\} \qquad (\text{with } \Lambda[0] = \varnothing)$$

as the spectrum of y. We shall say that a sequence $\{y_\nu\}$, $\nu = 1, 2,...$ from $V_n(\mathbb{R})$ is a U_ℓ-sequence, a U_r-sequence, a U-sequence, or a V-sequence if the corresponding sequence of spectral sets $\Lambda[y_\nu]$, $\nu = 1, 2,...$ satisfies

$$\limsup \Lambda[y_\nu] = -\infty,$$

$$\liminf \Lambda[y_\nu] = +\infty,$$

$$\liminf\{|\lambda| : \lambda \in \Lambda[y_\nu]\} = +\infty,$$

or

$$\sup\left\{|\lambda| : \lambda \in \bigcup_{\nu=1}^{\infty} \Lambda[y_\nu]\right\} < +\infty,$$

respectively.

Through a general sequence $\{y_\nu\}$ from $V_n(\mathbb{R})$ need not be a U_ℓ, U_r, U, or a V-sequence we may always find a subsequence of $\{y_\nu\}$ (which we continue to call $\{y_\nu\}$) that may be decomposed in the form

$$y_\nu = u_\nu + v_\nu, \qquad \nu = 1, 2, \ldots$$

where $\{u_\nu\}$ and $\{v_\nu\}$ are U and V-sequences from $V_{n_1}(\mathbb{R})$ and $V_{n_2}(\mathbb{R})$, respectively, with $n_1 + n_2 \leqslant n$ (cf. [8, pp. 82–83]). Furthermore, after again passing to a subsequence, if necessary, the U-sequence $\{u_\nu\}$ from $V_{n_1}(\mathbb{R})$ may be further decomposed in the form

$$u_\nu = \ell_\nu + r_\nu, \qquad \nu = 1, 2, \ldots$$

where $\{\ell_\nu\}$ and $\{r_\nu\}$ are U_ℓ and U_r-sequences from $V_{m_1}(\mathbb{R})$ and $V_{m_2}(\mathbb{R})$, respectively, with $m_1 + m_2 \leqslant n_1$.

We now establish a series of preparatory lemmas which deals with the important convergence properties of these types of sequences of exponential sums. We shall show that some subsequence of a $\|\ \|_{p,\sigma}$-bounded V-sequence converges to an exponential sum v and that some subsequence of a $\|\ \|_{p,\sigma}$-bounded U-sequence converges uniformly to zero on compact subsets of the interval (t_ℓ, t_r) where $t_\ell = \max L_{m_1,\sigma}$ and $t_r = \min R_{m_2,\sigma}$ for some choice of m_1 and m_2. We deal first with the convergence of V-sequences.

LEMMA 1. *Let $1 \leqslant p, q \leqslant \infty$, let σ be given, let n be a positive integer, let S be a bounded subset of $\mathbb{R}$, and let $I = [\alpha, \beta)$ be an interval containing at least n distinct points from D_σ. Then there is a positive constant M such that*

$$\| D^i y \|_{q,\sigma} \leqslant M \cdot \| y \cdot \chi(I; -)\|_{p,\sigma}, \qquad i = 0, 1, \ldots, n-1 \tag{5}$$

whenever $y \in V_n(S)$. (Henceforth $\chi(I; -)$ denotes the characteristic function of the set I.)

Proof. Given $y \in V_n(\mathbb{R})$ we can find $\mathbf{b} = (b_1, \ldots, b_n)$ and $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$ from $\mathbb{R}^n$ such that (1) and the initial conditions

$$D^{i-1} y(0) = b_i, \qquad i = 1, \ldots, n \tag{6}$$

hold. We let $Y_n(\mathbf{b}, \mathbf{y}, -)$ denote the solution of (1) which satisfies (6) noting that $Y_n(\mathbf{b}, \boldsymbol{\lambda}, -)$ depends continuously on $\mathbf{b}, \boldsymbol{\lambda}$ (cf. [5, pp. 21, 75–76]). Using the well known result that $Y_n(\mathbf{b}, \boldsymbol{\lambda}, -)$ has at most $n - 1$ real zeros unless $Y_n(\mathbf{b}, \boldsymbol{\lambda}, -) \equiv 0$ (cf. [15, p. 48, #75] or [21, p. 112, Theorem 2.1]) we conclude that

$$\| Y_n(\mathbf{b}, \boldsymbol{\lambda}, -) \cdot \chi(I; -)\|_{p,\sigma} = 0$$

if and only if $\mathbf{b} = \mathbf{0}$. Hence, if we restrict $\mathbf{b}$ to the surface ∂B^n of the unit ball in $\mathbb{R}^n$ we may define

$$F_i(\mathbf{b}, \boldsymbol{\lambda}) = \| D^i Y_n(\mathbf{b}, \boldsymbol{\lambda}, -)\|_{q,\sigma} / \| Y_n(\mathbf{b}, \boldsymbol{\lambda}, -) \cdot \chi(I; -)\|_{p,\sigma}, \quad i = 0, 1,..., n-1.$$

By enlarging S, if necessary, we may assume that S is compact. Since F_i is then a continuous function on the compact set $\partial B^n \times S^n$ there exists a constant $M_i > 0$ such that

$$F_i(\mathbf{b}, \boldsymbol{\lambda}) \leqslant M_i \qquad \text{for all} \quad \mathbf{b} \in \partial B^n, \quad \boldsymbol{\lambda} \in S^n, \quad i = 0, 1,..., n-1.$$

We then choose

$$M = \max\{M_0, M_1, ..., M_{n-1}\}$$

noting that (5) then holds for all $y \in V_n(S)$. ∎

LEMMA 2. *Let n be a positive integer, let $1 \leqslant p \leqslant \infty$, let σ be given, let D_σ contain at least n distinct points, and let S be a compact subset of $\mathbb{R}$. If $\{v_\nu\}$ is a $\| \ \|_{p,\sigma}$-bounded sequence from $V_n(S)$, then there exists a subsequence of $\{v_\nu\}$ which $\| \ \|_{p,\sigma}$-converges to some $v \in V_n(S)$.*

Proof. Let $\{\mathbf{b}_\nu\}$ and $\{\boldsymbol{\lambda}_\nu\}$ be chosen from $\mathbb{R}^n$ and S^n, respectively, so that

$$y_\nu = Y_n(\mathbf{b}_\nu, \boldsymbol{\lambda}_\nu, -), \qquad \nu = 1, 2,...$$

where Y_n is as in the proof of Lemma 1. By passing to a subsequence, if necessary, we may assume that $\{\boldsymbol{\lambda}_\nu\}$ converges to some $\boldsymbol{\lambda}^*$ in the compact set S^n. By Lemma 1, the $\| \ \|_{p,\sigma}$-boundedness of $\{y_\nu\}$ implies that $\| \ \|_{\infty,\sigma}$-boundedness of $\{D^i y_\nu\}$, $i = 0, 1,..., n-1$. Hence $\{\mathbf{b}_\nu\}$ is bounded, and by again passing to a subsequence, if necessary, we may assume that $\{\mathbf{b}_\nu\}$ converges to some $\mathbf{b}^* \in \mathbb{R}^n$. Since Y_n depends continuously on its parameters we see that

$$y = Y_n(\mathbf{b}^*, \boldsymbol{\lambda}^*, -)$$

is the uniform and hence also the $\| \ \|_{p,\sigma}$-limit of $\{y_\nu\}$. ∎

The following two lemmas deal with the decay of U_ℓ and U_r-sequences.

LEMMA 3. *Let n be a positive integer, let $\epsilon > 0$, let $t_1 < \cdots < t_n$, let*

$$\tau = (1/4) \cdot \min\{\epsilon, t_2 - t_1, ..., t_n - t_{n-1}\}, \tag{7}$$

and for each $\nu = 1, 2,...$ let $T_\nu = \{t_{i\nu}: i = 1,..., n\}$ be a set of n points such that

$$t_i - \tau \leqslant t_{i\nu} \leqslant t_i + \tau, \qquad i = 1,..., \nu. \tag{8}$$

If $\{\ell_\nu\}$ *is a* U_ℓ*-sequence from* $V_n(-\infty, 0)$ (*Henceforth we shall use* $V_n(a, b)$ *in place of* $V_n((a, b))$ *to avoid using double parentheses.*) *such that* $\{\ell_\nu(t_{i\nu})\}$ *is bounded for each* $i = 1,\ldots, n$, *then* $\{\ell_\nu\}$ *is uniformly bounded on* $[t_n + \tau, +\infty)$ *and uniformly converges to zero on* $[t_n + \epsilon, +\infty)$. *Analogously, if* $\{r_\nu\}$ *is a* U_r*-sequence from* $V_n(0, +\infty)$ *such that* $\{r_\nu(t_{i\nu})\}$ *is bounded for each* $i = 1,\ldots, n$, *then* $\{r_\nu\}$ *is uniformly bounded on* $(-\infty, t_1 - \tau]$ *and uniformly converges to zero on* $(-\infty, t_1 - \epsilon]$.

Proof. We give a proof for the case where the sequence under consideration is a U_ℓ-sequence; the proof for the U_r-sequence follows by "reflection."

The lemma clearly holds when $n = 1$ and so we now assume that $n \geqslant 2$. In proving this lemma we lose no generality in passing to a subsequence of $\{\ell_\nu\}$ whenever it is convenient to do so. For example, if $\{\ell_\nu\}$ is not uniformly bounded on $[t_n + \tau, +\infty)$, then (by passing to a subsequence, if necessary) we can find points t_ν^* from $[t_n + \tau, +\infty)$ such that $\lim |\ell_\nu(t_\nu^*)| = +\infty$. It follows that every subsequence fails to be uniformly bounded on $[t_n + \tau, +\infty)$. Consequently, we may assume that each term of the sequence has the parametric representation

$$\ell_\nu(t) = \sum_{i=1}^{\ell} \sum_{j=1}^{n_i} a_{ij\nu} t^{j-1} \exp(\lambda_{i\nu} \cdot t)$$

where $n_1 + \cdots + n_\ell = n$, $\lambda_{1\nu} < \cdots < \lambda_{\ell\nu}$, and $a_{i,n_i,\nu} \neq 0$ for $i = 1,\ldots, \ell$. Since the functions

$$t^{j-1} \exp(\lambda_{i\nu} \cdot t), \qquad i = 1,\ldots, \ell, \quad j = 1,\ldots, n_i \tag{9}$$

form a Haar System [4, pg. 74] for each $\nu = 1, 2,\ldots$ we can form a linear combination $\ell_{i\nu}$ of the functions (9) such that

$$\ell_{i\nu}(t_{j\nu}) = \delta_{ij}, \qquad j = 1,\ldots, n$$

for each $\nu = 1, 2,\ldots$ and for each $i = 1,\ldots, n$. (Here δ_{ij} denotes the familiar Kronecker symbol.) If we set $c_{i\nu} = \ell_\nu(t_{i\nu})$, $i = 1,\ldots, n$ we see that each ℓ_ν has the unique representation

$$\ell_\nu(t) = \sum_{i=1}^{n} c_{i\nu} \ell_{i\nu}(t)$$

where, according to the hypotheses of the lemma, $\{c_{i\nu}\}$ is a bounded sequence for each $i = 1,\ldots, n$.

To complete the proof of the lemma it is sufficient to show that each of the sequences $\{\ell_{i\nu}\}$, $i = 1,\ldots, n$ is uniformly bounded on $[t_n + \tau, +\infty)$ and uniformly converges to zero on $[t_n + \epsilon, +\infty)$. Let $1 \leqslant k \leqslant n$. Using

Rolle's theorem and the fact that $\ell_{k\nu}(+\infty) = 0$ for $\nu = 1, 2, \ldots$ and adopting the notation $t_{n+1,\nu} \equiv +\infty$ for $\nu = 1, 2, \ldots$ we see that the derivative $\ell'_{k\nu}$ has at least one zero in each of the $n - 1$ open intervals

$$\begin{aligned} &(t_{j\nu}, t_{j+1,\nu}), \qquad 1 \leqslant j \leqslant k - 2 \\ &(t_{k-1,\nu}, t_{k+1,\nu}) \\ &(t_{j\nu}, t_{j+1,\nu}), \qquad k + 1 \leqslant j \leqslant n. \end{aligned} \tag{10}$$

Since $\ell'_{k\nu} \in V_n(\mathbb{R})$ and $\ell'_{k\nu} \not\equiv 0$ it follows that $\ell'_{k\nu}$ can have no more than $n - 1$ distinct real zeros. Thus, $\ell'_{k\nu}$ has exactly $n - 1$ real zeros, one in each of the $n - 1$ intervals (10). By [9, Theorem 1] there exists a nonincreasing envelope function

$$\gamma_n \colon [0, +\infty) \to (0, 1] \tag{11}$$

with

$$\gamma_n(t) \downarrow 0 \qquad \text{as} \quad t \to +\infty$$

such that the inequality

$$|y(t)| \leqslant \max\{|y(s)| \colon 0 \leqslant s \leqslant \Gamma_n/\alpha\} \cdot \gamma_n(\alpha t), \qquad t \geqslant 0$$

holds for every $y \in V_n(-\infty, -\alpha)$ where $\alpha > 0$ and

$$\Gamma_n = \sup\{t \geqslant 0 \colon \gamma_n(t) = 1\}.$$

Thus, after passing to a subsequence, if necessary, we may assume that each term of the sequence $\{\ell_{k\nu}\}$ assumes its maximum modulus relative to the interval $[t_k - 2\tau, +\infty)$ on the subinterval $[t_k - 2\tau, t_k - \tau]$. Using the above information on the zeros of each $\ell'_{k\nu}$, the location of the maximum of each $\ell_{k\nu}$ relative to the interval $[t_k - 2\tau, +\infty)$, and the fact that each $\ell_{k\nu}$ assumes the value 1 at the point $t_{k\nu} \in [t_k - \tau, t_k + \tau]$, we see that on $[t_k + \tau, t_k + 2\tau]$ each $\ell_{k\nu}$ is positive, strictly decreasing, and bounded by 1. Using the envelope function of (11) once again, we see that after passing to yet another subsequence, if necessary, we may assume that the sequence $\{\ell_{k\nu}\}$ is uniformly bounded on $[t_k + \tau, +\infty)$ and hence also on $[t_n + \tau, +\infty)$ and that $\{\ell_{k\nu}\}$ uniformly converges to zero on $[t_n + \epsilon, +\infty)$. ∎

LEMMA 4. *Let n be a positive integer, let σ be given, let $1 \leqslant p \leqslant \infty$, let $t_1 < \cdots < t_n$ be chosen from D_σ, let $\epsilon > 0$ be arbitrarily chosen, and let*

$$u_\nu = \ell_\nu + r_\nu, \qquad \nu = 1, 2, \ldots$$

be a $\| \ \|_{p,\sigma}$-bounded U-sequence from $V_n(\mathbb{R})$ where $\{\ell_\nu\}$ is a U_ℓ-sequence from $V_{n_1}(\mathbb{R})$, where $\{r_\nu\}$ is a U_r-sequence from $V_{n_2}(\mathbb{R})$, and where $n_1 + n_2 \leqslant n$. Then $\{\ell_\nu\}$ uniformly converges to zero on $[t_{n_1} + \epsilon, +\infty)$ and $\{r_\nu\}$ uniformly converges to zero on $(-\infty, t_{n-n_2+1} - \epsilon]$. (We assume $t_0 = 0$ and $t_{n+1} = 1$.)

Proof. As in the proof of the preceding lemma, we lose no generality in passing to a subsequence of $\{u_\nu\}$ whenever it is convenient to do so. Therefore, we may assume that $\ell_\nu \in V_{n_1}(-\infty, 0)$, that $r_\nu \in V_{n_2}(0, +\infty)$, that ℓ_ν has full order n_1, and that r_ν has full order n_2, $\nu = 1, 2, \ldots$, and since Lemma 3 covers the case where n_1 or n_2 vanishes we may assume that n_1, $n_2 \geqslant 1$.

Let τ be defined as in (7). We next determine points $t_{i\nu}$, $i = 1, \ldots, n$ satisfying (8) and a constant $B > 0$ such that

$$| u_\nu(t_{i\nu})| \leqslant B, \qquad i = 1, \ldots, n, \quad \nu = 1, 2, \ldots . \tag{12}$$

Indeed, if $p = \infty$ we simply take $t_{i\nu} = t_i$, $i = 1, \ldots, n$ and let B be a $\| \ \|_{\infty,\sigma}$-bound on $\{u_\nu\}$. When $p < \infty$ more work is required. Assume $p < \infty$ and let $N_i = (t_i - \tau, t_i + \tau)$ noting that since $t_i \in D_\sigma$ the measure $M_i = \lambda_\sigma(N_i)$ of N_i is positive, $i = 1, \ldots, n$. Let $M = \min\{M_i : i = 1, \ldots, n\}$, let

$$C = \sup_\nu \| u_\nu \|_{p,\sigma}^p , \tag{13}$$

and let $B = (2C/M)^{1/p}$. Then for each $i = 1, \ldots, n$, and $\nu = 1, 2, \ldots$ there exists $t_{i\nu} \in N_i$ such that (12) holds. Indeed, if this is not the case we would have

$$\begin{aligned} \| u_\nu \|_{p,\sigma}^p &\geqslant \int_{N_i} | u_\nu(t)|^p \, d\lambda_\sigma \\ &\geqslant B^p \cdot \int_{N_i} d\lambda_\sigma \\ &\geqslant (2C/M) \cdot M_i \\ &> C \end{aligned}$$

in contradiction to (13).

Let $X_\nu = \{t_{1\nu}, \ldots, t_{n_1,\nu}\}$, let $Y_\nu = \{t_{n-n_2+1,\nu}, \ldots, t_{n,\nu}\}$, and let $Z_\nu = X_\nu \cup Y_\nu$, $\nu = 1, 2, \ldots$. Using (12), we see that

$$\| u_\nu \|_{Z_\nu} \leqslant B, \qquad \nu = 1, 2, \ldots \tag{14}$$

where we adopt the notation $\| \phi \|_X = \max\{| \phi(t)| : t \in X\}$.

Suppose for the moment that $\{\| \ell_\nu \|_{X_\nu}\}$ is bounded. Using Lemma 3, we conclude that $\{\| \ell_\nu \|_{Z_\nu}\}$ is bounded and, in conjunction with the triangle inequality, we infer that $\{\| r_\nu \|_{Z_\nu}\}$ is also bounded. A second application of Lemma 3 enables us to conclude that $\{\ell_\nu\}$ uniformly converges to zero on $[t_{n_1} + \epsilon, +\infty)$ and that $\{r_\nu\}$ uniformly converges to zero on $(-\infty, t_{n-n_2+1} - \epsilon]$. Thus, to complete the proof of the lemma we need only to show that $\{\| \ell_\nu \|_{X_\nu}\}$ is indeed bounded. Since $\{r_\nu(t_{i\nu})/\| r_\nu \|_{Y_\nu}\}$ is bounded for each $i = n - n_2 + 1, \ldots, n$ we may use Lemma 3 to conclude that

$$\| r_\nu \|_{X_\nu} \leqslant (1/2) \cdot \| r_\nu \|_{Y_\nu} \tag{15}$$

for all sufficiently large ν. Similarly,

$$\| \ell_\nu \|_{Y_\nu} \leqslant (1/2) \cdot \| \ell_\nu \|_{X_\nu} \tag{16}$$

for all sufficiently large ν. Using the triangle inequality, (14), (15), and (16), we find that

$$\begin{aligned}
\| \ell_\nu \|_{X_\nu} &\leqslant \| \ell_\nu + r_\nu \|_{X_\nu} + \| r_\nu \|_{X_\nu} \\
&\leqslant \| \ell_\nu + r_\nu \|_{Z_\nu} + (1/2) \cdot \| r_\nu \|_{Y_\nu} \\
&\leqslant B + (1/2) \cdot \{\| \ell_\nu + r_\nu \|_{Y_\nu} + \| \ell_\nu \|_{Y_\nu}\} \\
&\leqslant B + (1/2) \cdot \{B + (1/2) \cdot \| \ell_\nu \|_{X_\nu}\}
\end{aligned}$$

for all sufficiently large ν. It follows that $\{\| \ell_\nu \|_{X_\nu}\}$ is bounded. ∎

Having concluded the proofs of these preparatory lemmas we can now present the following two lemmas which together with Lemma 2 characterize the important properties of U and V sequences which are critical to the proof of our existence theorem.

LEMMA 5. *Let $1 \leqslant p \leqslant \infty$, let n be a positive integer, let σ be given, let D_σ contain at least n distinct points, and let $\{u_\nu + v_\nu\}$, $\nu = 1, 2,\ldots$ be a $\| \ \|_{p,\sigma}$-bounded sequence from $V_n(\mathbb{R})$ where $\{u_\nu\}$ is a U-sequence and $\{v_\nu\}$ is a V-sequence. Then the component sequences $\{u_\nu\}$, $\{v_\nu\}$ are $\| \ \|_{p,\sigma}$-bounded.*

Proof. It is sufficient to consider the case where

$$u_\nu = \ell_\nu + r_\nu, \qquad \nu = 1, 2,\ldots,$$

where ℓ_ν has order $n_1 \geqslant 0$, where r_ν has order $n_2 \geqslant 0$, where v_ν has order $n - n_1 - n_2$, and where $n > n_1 + n_2 > 0$.

Let B be a $\| \ \|_{p,\sigma}$-bound for $\{u_\nu + v_\nu\}$, let $t_1 < \cdots < t_n$ be n points of D_σ, and let $I = [\alpha, \beta)$ where $\alpha = 0$ if $n_1 = 0$ and $\alpha = (t_{n_1} + t_{n_1+1})/2$ if $n_1 > 0$, and where $\beta > 1$ if $n_2 = 0$ and $\beta = (t_{n-n_2} + t_{n-n_2+1})/2$ if $n_2 > 0$. By construction, I contains at least $n - n_1 - n_2 > 0$ points of D_σ. By Lemma 1, there is an $M > 0$ such that

$$\| v_\nu \|_{p,\sigma} \leqslant M \cdot \| v_\nu \cdot \chi(I; -)\|_{p,\sigma}, \qquad \nu = 1, 2,\ldots, \tag{17}$$

and, using Lemma 4, we see that the inequality

$$\| u_\nu \cdot \chi(I; -)\|_{p,\sigma} \leqslant \| u_\nu \|_{p,\sigma}/(2 \cdot M) \tag{18}$$

holds for all sufficiently large ν. In conjunction with the triangle inequality, the inequalities (17) and (18) yield

$$\begin{aligned} \| u_\nu \|_{p,\sigma} &\leqslant \| u_\nu + v_\nu \|_{p,\sigma} + \| v_\nu \|_{p,\sigma} \\ &\leqslant B + M \cdot \| v_\nu \cdot \chi(I; -)\|_{p,\sigma} \\ &\leqslant B + M \cdot \{\|(u_\nu + v_\nu) \cdot \chi(I; -)\|_{p,\sigma} + \| u_\nu \cdot \chi(I; -)\|_{p,\sigma}\} \\ &\leqslant B + M \cdot \{B + \| u_\nu \|_{p,\sigma}/(2M)\} \\ &= B \cdot (M + 1) + \| u_\nu \|_{p,\sigma}/2 \end{aligned}$$

for all sufficiently large ν. It follows that $\{u_\nu\}$ and $\{v_\nu\}$ are $\| \ \|_{p,\sigma}$-bounded. ∎

LEMMA 6. *Let $1 \leqslant p \leqslant \infty$, let n be a positive integer, let σ be given, let D_σ contain at least n distinct points, and let $\{u_\nu\}$ be a $\| \ \|_{p,\sigma}$-bounded U-sequence from $V_n(\mathbb{R})$ with $u_\nu = \ell_\nu + r_\nu$, $\nu = 1, 2,\ldots$, where $\{\ell_\nu\}$ is a U_ℓ-sequence from $V_{n_1}(\mathbb{R})$, where $\{r_\nu\}$ is a U_r-sequence from $V_{n_2}(\mathbb{R})$, and where $n_1 + n_2 \leqslant n$. Then for some subsequence of $\{u_\nu\}$ (which we continue to denote by $\{u_\nu\}$) there exists a real valued function u having support $L_{n_1,\sigma} \cup R_{n_2,\sigma}$ such that the inequality*

$$\underline{\lim} \, \| f + u_\nu \|_{p,\sigma} \geqslant \| f + u \|_{p,\sigma} \tag{19}$$

holds for all $f \in \mathscr{L}_{p,\sigma}$.

Proof. Since we can again pass to a subsequence whenever it is convenient to do so we may assume that $\{\| u_\nu \|_{p,\sigma}\}$ has a finite limit. Since $\{u_\nu\}$ is $\| \ \|_{p,\sigma}$-bounded, it follows that $\{u_\nu(s)\}$ is bounded for each $s \in L_{n_1,\sigma} \cup R_{n_2,\sigma}$ and after again passing to a subsequence, if necessary, we may assume these sequences converge. We then define

$$\begin{aligned} u(t) &= \lim_{\nu \to \infty} u_\nu(t), \text{ if } t \in L_{n_1,\sigma} \cup R_{n_2,\sigma} \\ &= 0, \text{ otherwise} \end{aligned}$$

If $L_{n_1,\sigma} \cup R_{n_2,\sigma} = D_\sigma$ we are done. If not, we set

$$\begin{aligned} t_\ell &= \max L_{n_1,\sigma}, \text{ if } n_1 > 0 \text{ and } L_{n_1,\sigma} \text{ contains } n_1 \text{ points} \\ &= \inf(D_\sigma \backslash L_{n_1,\sigma}), \text{ otherwise} \end{aligned}$$

and

$$\begin{aligned} t_r &= \min R_{n_2,\sigma}, \text{ if } n_2 > 0 \text{ and } R_{n_2,\sigma} \text{ contains } n_2 \text{ points} \\ &= \sup (D_\sigma \backslash R_{n_2,\sigma}), \text{ otherwise,} \end{aligned}$$

and for each ϵ such that $0 < \epsilon < (t_r - t_\ell)/2$ we set

$$\begin{aligned} i_\epsilon &= 0, && \text{if} \quad n_1 = 0 \\ &= t_\ell + \epsilon, && \text{otherwise}, \\ j_\epsilon &= 1 + \epsilon, && \text{if} \quad n_2 = 0 \\ &= t_r - \epsilon, && \text{otherwise}, \end{aligned}$$

and

$$I_\epsilon = [i_\epsilon, j_\epsilon).$$

Let $f \in L_{p,\sigma}$. Using Lemma 4, we conclude that $\{u_\nu\}$ uniformly converges to u on $I_\epsilon \cup L_{n_1,\sigma} \cup R_{n_2,\sigma}$. Hence,

$$\begin{aligned} \lim \| f + u_\nu \|_{p,\sigma} &\geqslant \underline{\lim} \, \|(f + u_\nu) \cdot \chi(I_\epsilon \cup L_{n_1,p,\sigma} \cup R_{n_2,p,\sigma}\, ; -)\|_{p,\sigma} \\ &= \|(f + u) \cdot \chi(I_\epsilon \cup L_{n_1,\sigma} \cup R_{n_2,\sigma}\, ; -)\|_{p,\sigma} \end{aligned}$$

and since $\epsilon > 0$ is arbitrarily small we conclude that (19) holds. Indeed, when $p < \infty$ then the Monotone Convergence Theorem [19, pp. 227–228] applied to the sequence $\{|(f + u) \cdot \chi(I_{1/\nu} \cup L_{n_1,\sigma} \cup R_{n_2,\sigma}\, ; -)|^p\}$ shows that (19) holds in the limit as $\nu \to +\infty$. When $p = \infty$ we must use the ∞-Assumption to conclude that (19) holds. ∎

Our method of proving the existence theorem requires that we be able to approximate each element from $P_{n,\sigma}(S)$ arbitrarily close by an exponential sum from $V_n(S)$. This is afforded us by the following lemma.

LEMMA 7. *Let* $1 \leqslant p \leqslant \infty$, *let* n *be a nonnegative integer, let* $S \subseteq \mathbb{R}$, *let* σ *be given, and let* $q \in P_{n,\sigma}(S)$. *Then there is a sequence* $\{y_\nu\}$ *from* $V_n(S)$ *such that* $\lim \| q - y_\nu \|_{p,\sigma} = 0$.

Proof. If $q \in P_{n,\sigma}(S)$ then

$$q(t) = v(t) + \ell(t) + r(t), \qquad t \in \mathbb{R}$$

where $v \in V_k(S)$, where ℓ has support $L_{n_1,\sigma}$, where r has support $R_{n_2,\sigma}$, and where $k + n_1 + n_2 \leqslant n$. We shall show how to construct sequences $\{\ell_\nu\}$, $\{r_\nu\}$ from $V_{n_1}(S)$, $V_{n_2}(S)$, respectively, such that $\lim \| \ell_\nu - \ell \|_{p,\sigma} = \lim \| r_\nu - r \|_{p,\sigma} = 0$ in which case

$$y_\nu = v + \ell_\nu + r_\nu, \qquad \nu = 1, 2, \ldots$$

is the desired sequence.

First of all, if $L_{n_1,\sigma} = \varnothing$ we take $\ell_\nu \equiv 0$, $\nu = 1, 2, \ldots$. Otherwise, we let $\{\lambda_\nu\}$ be chosen from $S \cap (-\infty, 0)$ with $\lambda_\nu \downarrow -\infty$. For each $\nu = 1, 2, \ldots$ we let $\lambda_\nu \in V_{n_1}(\{\lambda_\nu\})$ be the unique exponential sum of lowest order for which

$$\ell_\nu(t) = \ell(t) \qquad \text{for all} \quad t \in L_{n_1,\sigma}.$$

Such a unique ℓ_ν exists since $L_{n_1,\sigma}$ has at most n_1 points and the functions

$$\exp(\lambda_\nu t),\ t\exp(\lambda_\nu t),\dots,\ t^{n_1-1}\exp(\lambda_\nu t)$$

form a Markoff system [4, p. 76] of order n_1 on $\mathbb{R}$. Let $t_\ell = \max L_{n_1,\sigma}$. If $t_\ell = 1$, in which case $L_{n_1,\sigma} = D_\sigma$, then $\ell_\nu = \ell$ for each ν and we are done. Otherwise, $t_\ell < 1$ and by Lemma 3 there exists a $B > 0$ such that

$$|\ell_\nu(t)| \leqslant B \qquad \text{for} \quad t \geqslant t_\ell, \quad \nu = 1, 2,\dots.$$

If $p = \infty$ then from the ∞-Assumption it follows that there is some $\epsilon > 0$ such that $t_\ell + \epsilon < \inf(D_\sigma \backslash L_{n_1,\sigma})$. By Lemma 3 $\{\ell_\nu\}$ uniformly converges to zero on $[t_\ell + \epsilon, +\infty)$ and thus on $D_\sigma \backslash L_{n_1,\sigma}$ so that $\lim \| \ell_\nu - \ell \|_{\infty,\sigma} = 0$. If $p < \infty$ we let $0 < \epsilon < 1 - t_\ell$ and we see by using Lemma 3 and the above bound B that

$$\begin{aligned}\overline{\lim} \| \ell_\nu - \ell \|_{p,\sigma} &\leqslant \overline{\lim}[\|(\ell_\nu - \ell)\cdot\chi((t_\ell, t_\ell + \epsilon]; -)\|_{p,\sigma} \\ &\quad + \|(\ell_\nu - \ell)\cdot\chi((t_\ell + \epsilon, 1 + \epsilon]; -)\|_{p,\sigma}] \\ &= \overline{\lim} \| \ell_\nu \cdot \chi((t_\ell, t_\ell + \epsilon]; -)\|_{p,\sigma} \\ &\leqslant B\cdot\lambda_\sigma((t_\ell, t_\ell + \epsilon])^{1/p}\end{aligned}$$

and since $\lambda_\sigma(t_\ell, t_\ell + \epsilon] \to 0$ as $\epsilon \to 0+$ we conclude that $\lim \| \ell_\nu - \ell \|_{p,\sigma} = 0$.

Analogously, a U_r-sequence $\{r_\nu\}$ from $V_{n_2}(S)$ can be constructed so that $\lim \| r_\nu - r \|_{p,\sigma} = 0$. ∎

In view of this lemma we see that no function from $P_{n,\sigma}(S) \backslash V_n(S)$ has a best $\|\ \|_{p,\sigma}$-approximation from $V_n(S)$. Example 1 is of this type. However, there are other functions which have no best $\|\ \|_{p,\sigma}$-approximations from $V_n(S)$.

EXAMPLE 2. Let σ be defined by (4) and let f be any real valued function on $\mathbb{R}$ such that

$$\begin{aligned} f(t) &= N + 1, & &\text{if} \quad t = 0 \\ &= -1, & &\text{if} \quad t = i/N, \quad i = 1,\dots, N.\end{aligned}$$

Then for each p, $1 \leqslant p \leqslant \infty$,

$$\inf\{\|f - y\|_{p,\sigma} : y \in V_1(\mathbb{R})\} = N^{1/p}$$

but there is no exponential sum y from $V_1(\mathbb{R})$ such that $\|f - y\|_{p,\sigma} = N^{1/p}$. ∎

This example clearly generalizes to $V_n(\mathbb{R})$, $n > 1$.

It should be clear from Lemma 7 and the following existence theorem that $P_{n,\sigma}(S)$ is the $L_{p,\sigma}$-closure of $V_n(S)$.

THEOREM 1. *Let $S \subseteq R$, let $1 \leqslant p \leqslant \infty$, let σ be given, and let n be a positive integer. If D_σ contains no more than n points then every $f \in \mathscr{L}_{p,\sigma}$ has a best $\| \ \|_{p,\sigma}$-approximation from $V_n(S)$. If D_σ contains at least $n+1$ points then every $f \in \mathscr{L}_{p,\sigma}$ has a best $\| \ \|_{p,\sigma}$-approximation from $P_{n,\sigma}(S)$ if and only if S is closed.*

Proof. If there are no more than n points in D_σ and $S \neq \varnothing$ then we can find a $y \in V_n(S)$ such that $\|f - y\|_{p,\sigma} = 0$. The exponential sum y is then a best $\| \ \|_{p,\sigma}$-approximation to f. If $S = \varnothing$ then $y \equiv 0$ is a best approximation.

Otherwise, let S be closed and let $\{q_\nu\}$ be a minimizing sequence from $P_{n,\sigma}(S)$, i.e.,

$$\lim \|f - q_\nu\|_{p,\sigma} = \inf\{\|f - q\|_{p,\sigma} : q \in P_{n,\sigma}(S)\}.$$

In view of Lemma 7 q_ν can be $\| \ \|_{p,\sigma}$-approximated to within $1/\nu$ by some exponential sum y_ν from $V_n(S)$, $\nu = 1, 2, \ldots$. Thus,

$$\lim \|f - y_\nu\|_{p,\sigma} = \lim \|f - q_\nu\|_{p,\sigma} .$$

After passing to a subsequence, if necessary, we may assume that $\{y_\nu\}$ has been decomposed in the form

$$y_\nu = v_\nu + u_\nu , \qquad \nu = 1, 2, \ldots$$

where $\{v_\nu\}$ is a V-sequence from $V_{n_1}(S)$ and $\{u_\nu\}$ is a U-sequence from $V_{n_2}(S)$ with $n_1 + n_2 \leqslant n$. Using Lemma 5, we see that $\{v_\nu\}$ and $\{u_\nu\}$ are $\| \ \|_{p,\sigma}$-bounded. Thus, after passing to another subsequence, if necessary, we may assume, using Lemma 2, that $\{v_\nu\}$ $\| \ \|_{p,\sigma}$-converges to some $v \in V_{n_1}(S)$ and that there exists, in view of Lemma 6, a u with support $L_{m_1,\sigma} \cup R_{m_2,\sigma}$ such that $m_1 + m_2 \leqslant n_2$ and

$$\underline{\lim} \| \phi - u_\nu \|_{p,\sigma} \geqslant \| \phi - u \|_{p,\sigma}$$

for each $\phi \in \mathscr{L}_{p,\sigma}$. Thus,

$$\begin{aligned} \inf\{\|f - q\|_{p,\sigma} : q \in P_{n,\sigma}(S)\} &= \lim \|f - q_\nu\|_{p,\sigma} \\ &= \lim \|f - y_\nu\|_{p,\sigma} \\ &= \lim \|f - v - u_\nu\|_{p,\sigma} \\ &\geqslant \|f - v - u\|_{p,\sigma} , \end{aligned}$$

i.e., $v + u$ is a best $\| \ \|_{p,\sigma}$-approximation to f from $P_{n,\sigma}(S)$.

Conversely, suppose that there is some $\lambda \notin S$ which is a limit point of S. By the choice of λ we see that

$$\inf\{\|f - q\|_{p,\sigma} : q \in P_{n,\sigma}(S)\} = 0.$$

If there is a $q_0 \in P_{n,\sigma}(S)$ such that $\|f - q_0\|_{p,\sigma} = 0$ then f and q_0 must agree at almost all the points of D_σ. The function q_0 can be expressed in the form

$$q_0(t) = y_0(t) + u(t), \qquad t \in \mathbb{R}$$

where $y_0 \in V_k(S)$, where u has support $L_{n_1,\sigma} \cup R_{n_2,\sigma}$, and where $k + n_1 + n_2 \leqslant n$. Since y_0 must agree with f at each point of $D_\sigma \backslash (L_{n_1,\sigma} \cup R_{n_2,\sigma})$, since there are at least $n + 1 - n_1 - n_2 > k$ such points, and since $f - y_0 \in V_{k+1}(S)$ it follows by a zero counting argument that $f \equiv y_0$. However, this is a contradiction since $\lambda \notin S$. Thus, f has no best $\| \ \|_{p,\sigma}$-approximation from $P_{n,\sigma}(S)$. ▌

We summarize in the following corollary some of the cases where a best approximation from $V_n(S)$ does exist.

COROLLARY. *Let $S \subseteq \mathbb{R}$, let $1 \leqslant p \leqslant \infty$, let σ be given, and let n be a positive integer. If any of the conditions*

(i) *D_σ has n or fewer points,*

(ii) *S is compact,*

(iii) *S is closed, S is bounded above, and σ is continuous at* 0,

(iv) *S is closed, S is bounded below, and σ is continuous at* 1, *or*

(v) *S is closed and σ is continuous at* 0 *ahd at* 1, *hold then each $f \in \mathscr{L}_{p,\sigma}$ has a best $\| \ \|_{p,\sigma}$-approximation from $V_n(S)$.*

Proof. If (i), (ii), (iii), (iv), or (v) holds then $P_{n,\sigma}(S) = V_n(S)$. (In the case of (i) the functions involved must be viewed as functions just on D_σ for the equality $P_{n,\sigma}(S) = V_n(S)$ to hold.) ▌

4. NONCOMPACT DOMAIN

We shall now expand on the approach presented in the previous section to study the existence of best approximations when D_σ is an unbounded subset of $[a, +\infty)$, $a \in \mathbb{R}$. After again effecting a translation we may assume that $D_\sigma \subseteq [0, +\infty)$, that $0 \in D_\sigma$, and that $\sup D_\sigma = +\infty$.

For a given σ and a given p, $1 \leqslant p \leqslant \infty$, not every exponential sum from $V_\infty(\mathbb{R})$ will necessarily belong to $\mathscr{L}_{p,\sigma}$, and for this reason we define the corresponding universal spectral set $U_{p,\sigma}$ to be the set of those $\lambda \in \mathbb{R}$ for which the exponential sum $y(t) = \exp(\lambda t)$ lies in $\mathscr{L}_{p,\sigma}$. Clearly, for a given σ and a given p, $U_{p,\sigma}$ is either an open interval of the form $(-\infty, a)$, $-\infty < a \leqslant +\infty$, the closure of such an open interval, or the empty set. Indeed, $U_{\infty,\sigma} = (-\infty, 0]$ for each σ. However, Table I illustrates the dependence of the universal spectral set on σ and on p for $1 \leqslant p < \infty$.

TABLE I

σ	$U_{p,\sigma}$, $1 \leqslant p < \infty$
$\sigma(t) = 0$, if $t \leqslant 0$ $= t$, if $t > 0$	$(-\infty, 0)$
$\sigma(t) = 0$, if $t \leqslant 0$ $= \exp(t) - 1$, if $t > 0$	$(-\infty, -p^{-1})$
$\sigma(t) = 0$, if $t \leqslant 0$ $= \sigma(n-1) + 1/n^2$, if $n - 1 < t \leqslant n$, $n = 1, 2, \ldots$	$(-\infty, 0]$
$\sigma(t) = 0$, if $t \leqslant 0$ $= 1 - 2^{-n}$, if $n - 1 < t \leqslant n$, $n = 1, 2, \ldots$	$(-\infty, p^{-1} \cdot \log 2)$
$\sigma(t) = 0$, if $t \leqslant 0$ $= 1 - \exp(-t^2)$, if $t > 0$	$(-\infty, \infty)$
$\sigma(t) = 0$, if $t \leqslant 0$ $= \exp(t^2) - 1$, if $t > 0$	$\varnothing$

The following lemma demonstrates the importance of the interior of the universal spectral set, $\operatorname{Int}(U_{p,\sigma})$.

LEMMA 8. *Let $1 \leqslant p \leqslant \infty$ and let σ be given. Then*

$$\bigcup_{n=1}^{\infty} P_{n,\sigma}(\operatorname{Int}(U_{p,\sigma})) \subseteq \mathscr{L}_{p,\sigma}.$$

Proof. It is enough to show that when $\lambda \in \operatorname{Int}(U_{p,\sigma})$ and n is a fixed nonnegative integer then $t^n \exp(\lambda t) \in \mathscr{L}_{p,\sigma}$. If $\lambda \in \operatorname{Int}(U_{p,\sigma})$ there is some $\mu > \lambda$ with $\mu \in \operatorname{Int}(U_{p,\sigma})$. Hence $t^n \exp[(\lambda - \mu)t] \to 0$ as $t \to \infty$. It follows that $t^n \exp(\lambda t) \in \mathscr{L}_{p,\sigma}$ with

$$\| t^n \exp(\lambda t)\|_{p,\sigma} \leqslant M \cdot \| \exp(\mu t)\|_{p,\sigma} < +\infty$$

where M is a uniform bound for the function $t^n \exp[(\lambda - \mu)t]$ on $[0, +\infty)$. ∎

We first extend Lemma 7 to this setting and then we present our existence theorem.

LEMMA 9. *Let $1 \leqslant p \leqslant \infty$, let n be a positive integer, let S be a subset of $\mathbb{R}$, let σ be given, and assume that $U_{p,\sigma} \neq 0$. Then for each $q \in P_{n,\sigma}(S)$ there is a sequence $\{y_\nu\}$ from $V_n(S)$ such that*

$$\lim \| q - y_\nu \|_{p,\sigma} = 0.$$

Proof. If $q \in P_{n,\sigma}(S)$ then

$$q(t) = v(t) + u(t), \qquad t \in \mathbb{R}$$

where $v \in V_k(S)$, where u has support $L_{m,\sigma}$, and where $k + m \leqslant n$. (Note that $R_{n,\sigma} = \varnothing$ since D is not bounded above.) We shall show how to construct a sequence $\{u_\nu\}$ from $V_m(S)$ such that $\lim \| u_\nu - u \|_{p,\sigma} = 0$ in which case

$$y_\nu = v + u_\nu, \qquad \nu = 1, 2, \ldots$$

is the desired sequence.

If $L_{m,\sigma} = \varnothing$ we take $u_\nu \equiv 0$, $\nu = 1, 2, \ldots$. Otherwise, S is not bounded below and since $U_{p,\sigma}$ is at least a semi-infinite interval we can choose $\{\lambda_\nu\}$ from $U_{p,\sigma} \cap (-\infty, 0) \cap S$ such that $\lambda_\nu \downarrow -\infty$. For each $\nu = 1, 2, \ldots$ we let $u_\nu \in V_m(\{\lambda_\nu\})$ be the unique exponential sum of lowest order which interpolates u on $L_{m,\sigma}$. Let $\epsilon > 0$ be arbitrarily chosen. We will show that $\| u_\nu - u \|_{p,\sigma} < \epsilon$ for all sufficiently large ν. Let $t_\ell = \max L_{m,\sigma}$. By Lemma 3 there is a $B > 0$ such that

$$| u_\nu(t)| \leqslant B \qquad \text{for} \quad t \geqslant t_\ell, \quad \nu = 1, 2, \ldots .$$

By [9, Theorem 1]

$$| u_\nu(t)| \leqslant B \cdot \gamma_m(-\lambda_\nu(t - t_\ell)), \qquad t \geqslant t_\ell$$

where γ_m is the envelope function of (11). Given any $\delta_1 > 0$ we have

$$\begin{aligned}
\| u_\nu - u \|_{p,\sigma} &= \| u_\nu \cdot \chi((t_\ell, +\infty); -)\|_{p,\sigma} \\
&\leqslant \| u_\nu \cdot \chi((t_\ell, t_\ell + \delta_1]; -)\|_{p,\sigma} \\
&\quad + \| u_\nu \cdot \chi((t_\ell + \delta_1; +\infty); -)\|_{p,\sigma} \\
&\leqslant B \cdot \| \chi((t_\ell, t_\ell + \delta_1]; -)\|_{p,\sigma} \\
&\quad + B \cdot \| \gamma_m(-\lambda_\nu(t - t_\ell)) \cdot \chi((t_\ell + \delta_1, +\infty); -)\|_{p,\sigma} .
\end{aligned}$$

Now as $\delta_1 \downarrow 0+$, the first term on the right of the above inequality goes to zero even when $p = \infty$ due to the ∞-Assumption. By [9, Theorem 1] $\gamma_m(-\lambda_\nu(t - t_\ell))$ uniformly converges to 0 on $(t_\ell + \delta_1, +\infty)$ as $\nu \to +\infty$ and so, if $p = \infty$ we can make the second term on the right of the above inequality less than $2\epsilon/3$ and the proof is finished.

If $p < \infty$, we note that for each $\delta_2 > 0$

$$\begin{aligned}
\| u_\nu - u \|_{p,\sigma} \leqslant \epsilon/3 &+ B \cdot \| \gamma_m(-\lambda_\nu(t - t_\ell)) \cdot \chi((t_\ell + \delta_1, t_\ell + \delta_1 + \delta_2]; -)\|_{p,\sigma} \\
&+ B \cdot \| \gamma_m(-\lambda_1(t - t_\ell)) \cdot \chi((t_\ell + \delta_1 + \delta_2, +\infty); -)\|_{p,\sigma}
\end{aligned}$$

where we use λ_1 in place of λ_ν in the third term on the right of the inequality since γ_m is nonincreasing. In view of [9, Lemma 3] it follows that the envelope function γ_m has the form

$$\gamma_m(t) = \pi(t) \cdot \exp(-t), \qquad t \geqslant \Gamma_m$$

where $\pi(t)$ is a polynomial of degree $m - 1$. Using this form of γ_m and Lemma 8, we see that the function $\gamma_m(-\lambda_1(t - t_\ell)) \in \mathscr{L}_{p,\sigma}$. Hence we can choose δ_2 so large that the third term on the right of the preceding inequality is less than $\epsilon/3$. With δ_1, δ_2 so fixed we can make use of the uniform convergence of $\gamma_m(-\lambda_\nu(t - t_\ell))$ to zero on $[t_\ell + \delta_1, +\infty)$ to conclude that the middle term on the right of the preceding inequality is less than $\epsilon/3$ for all sufficiently large ν. ∎

In view of the preceding lemma we once again see that no $f \in P_{n,\sigma}(S) \backslash V_n(S)$ has a best $\| \ \|_{p,\sigma}$-approximation from $V_n(S)$. We saw in example 2 for compact D_σ that there are other functions for which no best $\| \ \|_{p,\sigma}$-approximation exists. The same, of course, is true in this setting as the following example illustrates.

EXAMPLE 3. Let σ be defined recursively such that

$$\begin{aligned} \sigma(t) &= 0, && \text{if } \ t \leqslant 0 \\ &= \sigma(n-1) + 1, && \text{if } \ n - 1 < t \leqslant n, \quad n = 1, 2, \dots . \end{aligned}$$

Thus, D_σ is the set of nonnegative integers. Let f be any function defined on $\mathbb{R}$ satisfying

$$f(i+1) \cdot f(i) < 0, \qquad i = 0, 1, 2, \dots$$

and

$$0 < |f(i+1)| < |f(i)|, \qquad i = 0, 1, 2, \dots .$$

Then for each $n = 1, 2, \dots$ there is no best $\| \ \|_{\infty,\sigma}$-approximation to f from $V_n(\mathbb{R})$. Moreover, for each n the function q_n given by

$$\begin{aligned} q_n(t) &= f(t), && \text{if } \ t = 0, 1, \dots, n-1 \\ &= 0, && \text{otherwise} \end{aligned}$$

is a best $\| \ \|_{\infty,\sigma}$-approximation to f from $P_{n,\sigma}(\mathbb{R})$. ∎

We now present the existence theorem.

THEOREM 2. *Let $1 \leqslant p \leqslant \infty$, let n be a nonnegative integer, let σ be given, and let S be a closed subset of $\mathbb{R}$. Furthermore, assume that $U_{p,\sigma} \neq \varnothing$. Then every $f \in \mathscr{L}_{p,\sigma}$ has a best $\| \ \|_{p,\sigma}$-approximation from $P_{n,\sigma}(S)$.*

Proof. Let $\{q_\nu\}$ be a minimizing sequence from $P_{n,\sigma}(S)$. Using Lemma 9, we can approximate each q_ν, $\nu = 1, 2,\ldots$ to within $1/\nu$ by an exponential sum y_ν from $V_n(S)$. For a suitably chosen subsequence of $\{y_\nu\}$ which we continue to call $\{y_\nu\} = v_\nu + \ell_\nu + r_\nu$, $\nu = 1, 2,\ldots$ where $\{v_\nu\}$ is a V-sequence from $V_{m_1}(S)$, where $\{\ell_\nu\}$ is a U_ℓ-sequence from $V_{m_2}(S)$, and where $\{r_\nu\}$ is a U_r-sequence from $V_{m_3}(S)$, with $m_1 + m_2 + m_3 \leqslant n$.

For each $\alpha > 0$ such that the interval $I_\alpha = [0, \alpha)$ contains at least n points of D_σ we define the function

$$\begin{aligned} \sigma_\alpha(t) &= \sigma(t), &\quad \text{if} \quad t \leqslant \alpha \\ &= \sigma(\alpha-), &\quad \text{if} \quad t > \alpha \end{aligned}$$

so that

$$\|f\|_{p,\sigma_2} = \|f \cdot \chi(I_\alpha\,;\, -)\|_{p,\sigma}\,.$$

For such an α we let $\beta > \alpha$ be chosen from D_σ such that $[\alpha, \beta)$ contains at least $m_3 + 1$ additional points of D_σ. (If $p = \infty$ and if there is an accumulation points to the right of α we shall choose β to be such an accumulation point so that σ_β will satisfy the ∞-Assumption.) Since $\{y_\nu\}$ is $\|\ \|_{p,\sigma}$-bounded, it is also $\|\ \|_{p,\sigma_\beta}$-bounded and hence by Lemmas 4, 5 the component sequences $\{v_\nu\}$, $\{\ell_\nu\}$, and $\{r_\nu\}$ are $\|\ \|_{p,\sigma_\beta}$-bounded. Thus, by Lemma 2 we may assume that some subsequence of $\{v_\nu\}$, $\|\ \|_{p,\sigma_\beta}$-converges to $v \in V_{m_1}(S)$. In view of the way β is chosen we see, as in Lemma 6, that there are functions ℓ, r with support L_{m_2,σ_β}, R_{m_3,σ_β}, respectively, so that

$$\begin{aligned} \|f - v - \ell\|_{p,\sigma_\alpha} &\leqslant \|f - v - \ell - r\|_{p,\sigma_\beta} \\ &\leqslant \underline{\lim}\, \|f - v - u_\nu\|_{p,\sigma_\beta} \\ &= \underline{\lim}\, \|f - v_\nu - u_\nu\|_{p,\sigma_\beta} \\ &\leqslant \lim \|f - y\|_{p,\sigma} \\ &= \lim \|f - q_\nu\|_{p,\sigma} \end{aligned}$$

for the given α. Since we may assume the functions v and ℓ are independent of α, this holds in the limit as $\alpha \to +\infty$. Thus

$$\|f - v - \ell\|_{p,\sigma} \leqslant \inf\{\|f - q\|_{p,\sigma} \colon q \in P_{n,\sigma}(S)\}. \tag{20}$$

Since $v + l \in P_{n,\sigma}(S)$ equality actually holds in (20) and the proof is complete. ∎

As is the case when D_σ is compact there are cases when $P_{n,\sigma}(S) = V_n(S)$, e.g., if S is bounded below or if σ is continuous at 0 then $P_{n,\sigma} = V_n(S)$ and every $f \in \mathscr{L}_{p,\sigma}$ has a best $\|\ \|_{p,\sigma}$-approximation from $V_n(S)$.

5. Completely Monotone Functions

Frequently, one wishes to model decay type data based on observations taken at uniformly or nonuniformly spaced time intervals, cf. [1, p. 272] and [6]. The uniform approximation of such discrete data falls within the scope of $\| \|_{\infty,\sigma}$-approximation, e.g., where σ is a step function with jumps at the points of observation. It may be that for a general decay function F there is no best $\| \|_{\infty,\sigma}$-approximation from $V_n(\mathbb{R})$ and that the enlarged approximating class $P_{n,\sigma}(\mathbb{R})$ is needed to obtain a best approximation. However, when the data agrees with a completely monotone function we shall see that there is a best approximation from $V_n(\mathbb{R})$ and that no better approximation is obtained from the class $P_{n,\sigma}(\mathbb{R})\backslash V_n(\mathbb{R})$.

A function F is said to be completely monotone on $[0, \infty]$ if

$$F \in C^\infty(0, \infty) \cap C[0, \infty]$$

and

$$(-1)^m F^{(m)}(t) \geqslant 0, \qquad 0 < t < \infty, \qquad m = 0, 1,\ldots .$$

The approximation of such functions by elements from $V_n(\mathbb{R})$ has been studied by Braess [3], by Kammler [11, 12] and by this author [14]. Indeed, in [14, Theorem 2] we have shown that there is a best uniform approximation from $V_n(\mathbb{R})$ to a given completely monotone function F on a closed subset T of $[0, \infty)$ assuming that $F(\infty) = 0$ if $\sup T = +\infty$.

The number of alternations of the error curve $\epsilon = F - Y$ is a standard criterion for determining whether an approximation Y to a function F is a best uniform approximation, cf. [1; 16; 17, Chapter 8]. We say that the error curve alternates at least m times on T provided there exist $m + 1$ points of T, $t_1 < \cdots < t_{m+1}$, such that

$$| \epsilon(t_i)| = \| \epsilon \|, \qquad i = 1,\ldots, m + 1$$

and

$$\epsilon(t_i) = -\epsilon(t_{i+1}), \qquad i = 1,\ldots, m.$$

We now present a sufficient condition based on the number of alternations of the error curve for a function from $P_{n,\sigma}(\mathbb{R})$ to be a best $\| \|_{\infty,\sigma}$-approximation to a given function, cf. [16, p. 158].

Theorem 3. *Let σ satisfy the ∞-Assumption, let F be continuous on D_σ, and let n be a positive integer. Let $q \in P_{n,\sigma}(\mathbb{R})$, i.e.,*

$$q(t) = Y(t) + \ell(t) + r(t), \qquad t \in \mathbb{R}$$

where $Y \in V_k(\mathbb{R})$, where ℓ has support $L_{k_1,\sigma}$, where r has support $R_{k_2,\sigma}$, and where $k + k_1 + k_2 \leqslant n$. Furthermore, let $\epsilon = F - q$ alternate at least $n + k$ times on $D_\sigma \backslash (L_{k_1,\sigma} \cup R_{k_2,\sigma})$ and let

$$\|\epsilon\|_{\infty,\sigma} = \|\epsilon \cdot \chi(D_\sigma \backslash (L_{k_1,\sigma} \cup R_{k_2,\sigma}); -)\|_{\infty,\sigma}.$$

Then q is a best $\| \ \|_{\infty,\sigma}$-approximation to F from $P_{n,\sigma}(\mathbb{R})$.

Proof. Let $q^*(t)$ be a best $\| \ \|_{\infty,\sigma}$-approximation to F from $P_{n,\sigma}(\mathbb{R})$ as guaranteed by Theorem 1 when D_σ is compact and Theorem 2 when D_σ is not compact. Then

$$q^*(t) = Y^*(t) + \ell^*(t) + r^*(t), \qquad t \in \mathbb{R}$$

where $Y^* \in V_{k^*}(\mathbb{R})$, where ℓ^* has support $L_{k_1^*,\sigma}$, where r^* has support $R_{k_2^*,\sigma}$, and where $k^* + k_1^* + k_2^* \leqslant n$. Let $t_\ell = \inf(D_\sigma \backslash L_{k_1,\sigma})$, $t_r = \sup(D_\sigma \backslash R_{k_2,\sigma})$, $t_\ell^* = \inf(D_\sigma \backslash L_{k_1^*,\sigma})$, $t_r = \sup(D_\sigma \backslash R_{k_2^*,\sigma})$, and denote the intervals $[t_\ell, t_r]$, $[t_\ell^*, t_r^*]$ by I and I^*, respectively. Then

$$\|(F - Y^*) \cdot \chi(I^*; -)\|_{\infty,\sigma} \leqslant \|(F - Y) \cdot \chi(I; -)\|_{\infty,\sigma},$$

so that $Y^* - Y$ is alternately nonnegative and nonpositive on at least

$$\begin{aligned} n + k + 1 - (k_1^* + k_2^*) &\geqslant n + k + 1 - (n - k^*) \\ &= k + k^* + 1 \end{aligned}$$

points of D_σ. Since $Y^* - Y \in V_{k^*+k}$ we see, using a standard zero counting argument, that $Y^* \equiv Y$. Consequently, since $I \cap I^* \neq \varnothing$,

$$\|F - q\|_{\infty,\sigma} = \|F - q^*\|_{\infty,\sigma}$$

and q is also a best $\| \ \|_{\infty,\sigma}$-approximation to F. ∎

COROLLARY. *Let σ be given as in the theorem, let F be continuous on D_σ, and let n be a positive integer. Let $Y \in V_n(\mathbb{R})$ have order k and let $\epsilon = F - Y$ alternate at least $n + k$ times on D_σ. Then Y is a unique best $\| \ \|_{\infty,\sigma}$-approximation to F from $V_n(\mathbb{R})$. Moreover, if Y has order n then Y is a unique best $\| \ \|_{\infty,\sigma}$-approximation to F from $P_{n,\sigma}(\mathbb{R})$.*

Proof. From the theorem we see that Y is a best $\| \ \|_{\infty,\sigma}$-approximation to F from $P_{n,\sigma}(\mathbb{R})$ and hence from $V_n(\mathbb{R})$ as well. The uniqueness results follow by standard zero counting arguments. ∎

THEOREM 4. *Let σ satisfy the ∞-Assumption, let F be a completely monotone function on* $[0, \infty]$ *with $F(\infty) = 0$ if $\sup D_\sigma = +\infty$, and let n*

be a positive integer. Then each best $\| \|_{\infty,\sigma}$*-approximation to* F *from* $P_{n,\sigma}(\mathbb{R})$ *lies in* $V_n(\mathbb{R})$, *i.e., each best approximation agrees (at least on* D_σ*) with an element from* $V_n(\mathbb{R})$.

Proof. If D_σ contains no more than $2n$ points then by the Corollary to [14, Theorem 1] there is an exponential sum Y from $V_n(\mathbb{R})$ which interpolates F at the points of D_σ, and so each best approximation from $P_{n,\sigma}(\mathbb{R})$ must agree with Y on D_σ.

If D_σ contains at least $2n + 1$ points and if $F \in V_n(R)$ then by [14, Theorem 2] there is a best approximation Y to F from $V_n(\mathbb{R})$ such that $F - Y$ alternates exactly $2n$ times on D_σ. By the above Corollary to Theorem 3, Y is a unique best approximation to F from $P_{n,\sigma}(\mathbb{R})$. ∎

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